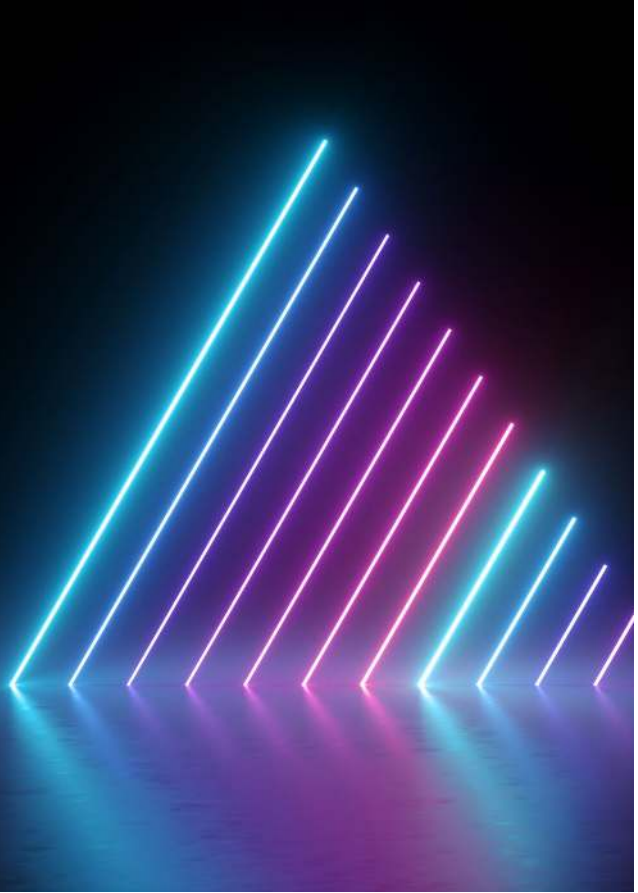


7.2 Eigenvalues and Eigenvectors

If A is a square matrix, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

■

It is not difficult to show (see Exercise 15) that p is an nth-degree polynomial and, consequently, has at most n distinct zeros, some of which might be complex. If λ is a zero of p, then, since $\det(A - \lambda I) = 0$, Theorem 6.17 on page 402 implies that the linear system defined by $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a solution with $\mathbf{x} \neq \mathbf{0}$. We wish to study the zeros of p and the nonzero solutions corresponding to these systems.

Definition 7.13 If p is the characteristic polynomial of the matrix A, the zeros of p are called **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ. ■

en comes from the
tive meaning
synonymous in
he word
. Each matrix has
or characteristic
corresponding
acteristic values

To determine the eigenvalues of a matrix, we can use the fact that

- λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Once an eigenvalue λ has been found, a corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ is determined by solving the system

- $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

EXERCISE SET 7.2

1. Compute the eigenvalues and associated eigenvectors of the following matrices.

a. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ e. $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ f. $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

1) a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$

$(2-\lambda)^2 = 2^2 - 2(2)(\lambda) + \lambda^2$

$\det(A - \lambda I) = (2-\lambda)(2-\lambda) - (-1)(-1)$

$= 4 - 4\lambda + \lambda^2 - 1$

$= \lambda^2 - 4\lambda + 3$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, 3$

For $\lambda = 1$

$\begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$R1 + R2$

$1 - 1 = 0 \quad -1 + 1 = 0$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$X_1 - X_2 = 0$

$X_1 = X_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1, 1)^t$

For $\lambda = 3$

$-(-1) - 1 = 1 - 1 = 0$

$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-X_1 - X_2 = 0$

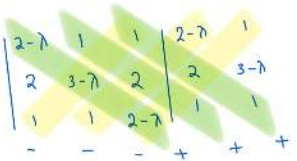
$X_1 = -X_2$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1, 1)^t$

$\lambda = 1$ eigen vector is $(1, 1)^t$

$\lambda = 3$ eigen vector is $(-1, 1)^t$

1) f) $A = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{bmatrix}$



$\det(A-\lambda I) = (2-\lambda)(3-\lambda)(2-\lambda) + (1)(2)(1) + (1)(2)(1) - (1)(3-\lambda)(1) - (2-\lambda)(2)(1) - (1)(2)(2-\lambda)$

$= (2-\lambda)^2(3-\lambda) + 2 + 2 - (3-\lambda) - (4-2\lambda) - (4-2\lambda)$

$= (4-4\lambda+\lambda^2)(3-\lambda) + 4 - 3 + \lambda - 4 + 2\lambda - 4 + 2\lambda$

$= (12 - 12\lambda + 3\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3) - 7 + 5\lambda$

$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$

$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$

$\lambda = 1, 5$

For $\lambda=1$

$$A-\lambda I=\begin{bmatrix}2-1 & 1 & 1\\ 2 & 3-1 & 2\\ 1 & 1 & 2-1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 1 & 1 & 1\end{bmatrix}\xrightarrow[-R_1+R_3]{-2R_1+R_2}\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$x_1+x_2+x_3=0$

$x_1=-x_2-x_3$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-x_2-x_3\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-x_2\\ x_2\\ 0\end{bmatrix}+\begin{bmatrix}-x_3\\ 0\\ x_3\end{bmatrix}$$

$$=x_2\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

eigen vectors

For $\lambda=5$

$$A-\lambda I=\begin{bmatrix}2-5 & 1 & 1\\ 2 & 3-5 & 2\\ 1 & 1 & 2-5\end{bmatrix}=\begin{bmatrix}-3 & 1 & 1\\ 2 & -2 & 2\\ 1 & 1 & -3\end{bmatrix}\xrightarrow{R1\leftrightarrow R3}\begin{bmatrix}1 & 1 & -3\\ 2 & -2 & 2\\ -3 & 1 & 1\end{bmatrix}\xrightarrow[3R_1+R_3]{-2R_1+R_2}\begin{bmatrix}1 & 1 & -3\\ 0 & -4 & 8\\ 0 & 4 & -8\end{bmatrix}\xrightarrow{-R_2+R_3}\begin{bmatrix}1 & 1 & -3\\ 0 & -4 & 8\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -3\\ 0 & -4 & 8\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$-4x_2+8x_3=0$

$\frac{4x_2}{4}=\frac{8x_3}{4}$

$x_2=2x_3$

$x_1+x_2-3x_3=0$

$x_1+2x_3-3x_3=0$

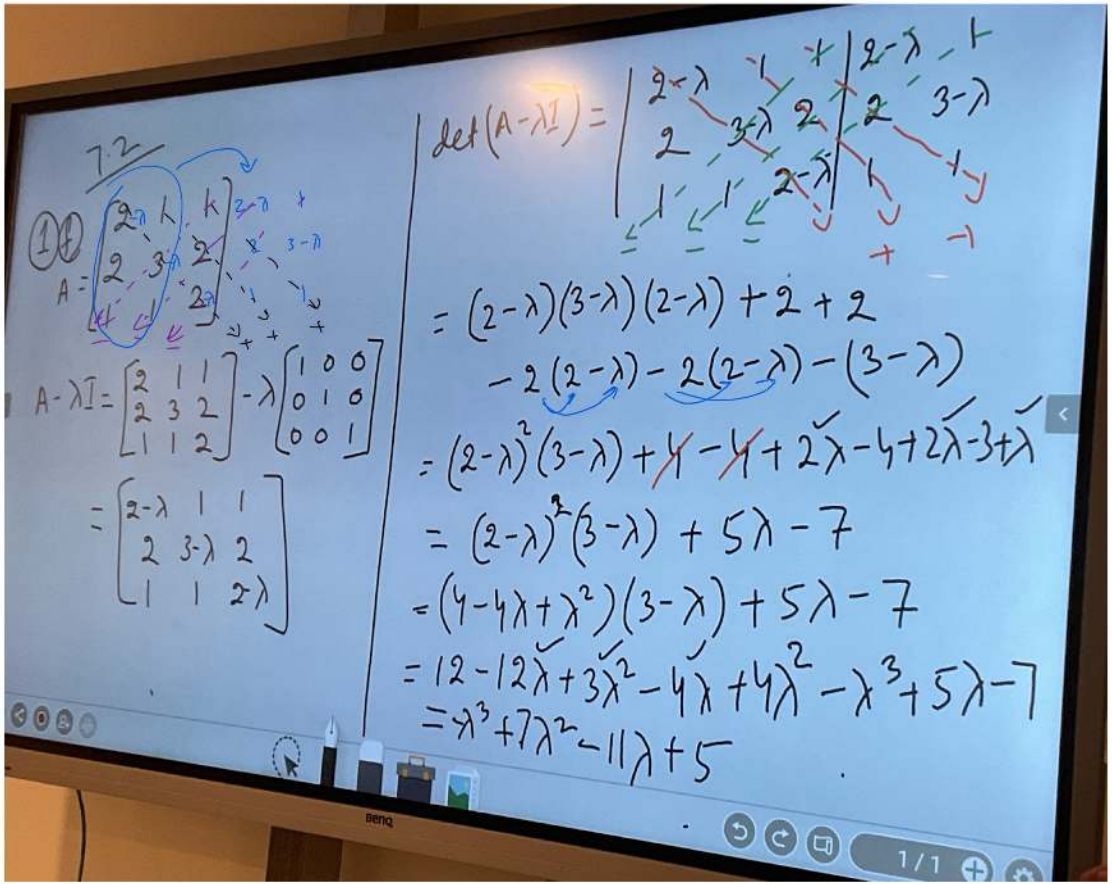
$x_1-x_3=0$

$x_1=x_3$

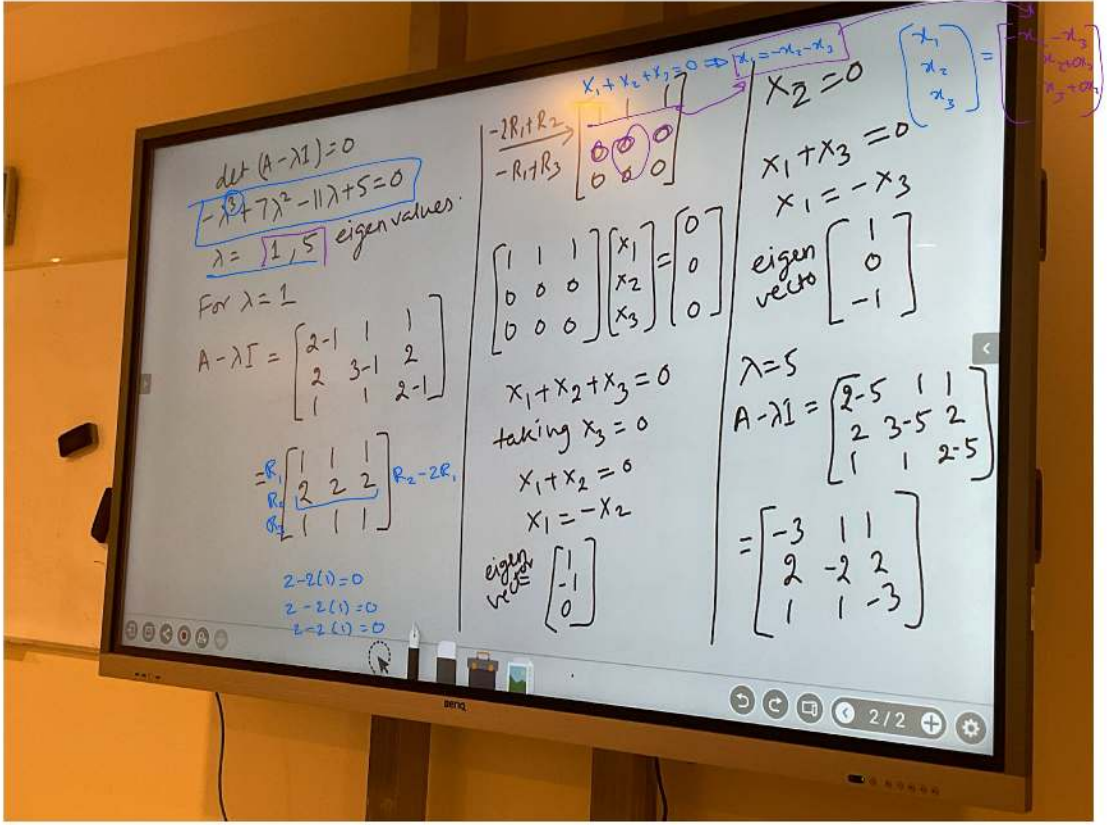
eigen vector $\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}=(1,2,1)^t$

$\lambda=1$ eigen vectors are $(-1,1,0)^t$, $(-1,0,1)^t$

$\lambda=5$ eigen vector is $(1,2,1)^t$



$\begin{matrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{matrix}$ $x_1 = 2$ $x_2 = 3$ $x_3 = 4$



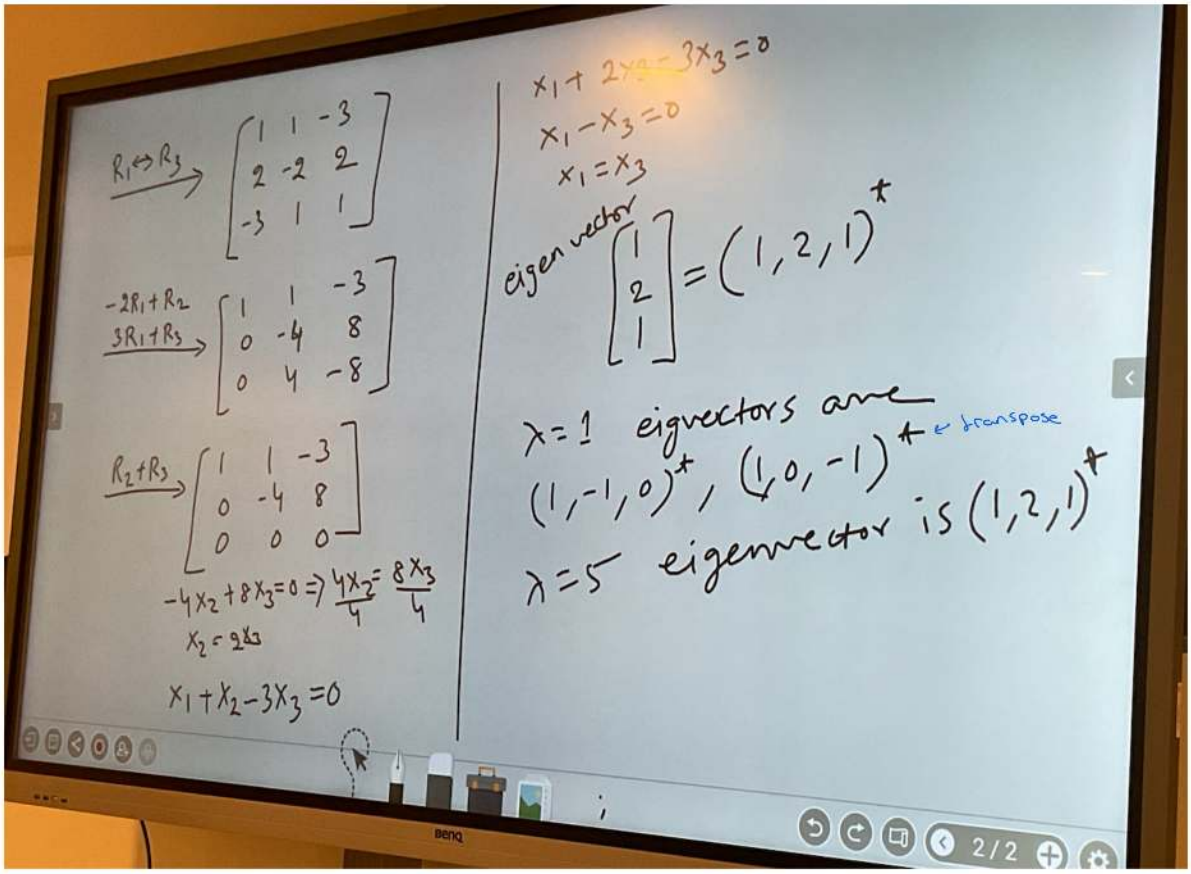
$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

eigen vectors

$x_1 + x_2 + x_3 = 0$

take $x_3 = 0$: $x_1 + x_2 = 0$, $x_1 = -x_2$, 1st eigen vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

take $x_2 = 0$: $x_1 + x_3 = 0$, $x_1 = -x_3$, 2nd eigen vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$


$R_1 \leftrightarrow R_3$
$\begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -3 & 1 & 1 \end{bmatrix}$
$-2R_1+R_2$
$3R_1+R_3$
$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix}$
R_2+R_3
$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$
$-4x_2+8x_3=0 \Rightarrow \frac{4x_2}{4} = \frac{8x_3}{4}$
$x_2 = 2x_3$
$x_1+x_2-3x_3=0$
$x_1+2x_2-3x_3=0$
$x_1-x_3=0$
$x_1=x_3$
eigenvector
$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = (1,2,1)^t$
$\lambda=1$ eigvectors are
$(1,-1,0)^t$, $(1,0,-1)^t$ ← transpose
$\lambda=5$ eigenvector is $(1,2,1)^t$
BenQ
2/2