Algorithms and Convergence

① Rate of Convergence for Sequences :

Formula : $|\alpha_n - \alpha| \le K|\beta_n|$, for large n, , $\beta_n = \frac{1}{n^p}$,

6. Find the rates of convergence of the following sequences as $n \to \infty$.

a. $\lim_{n\to\infty} \sin\frac{1}{n} = 0$ ($\frac{1}{n} \to x$)

to solve Q use ⇒

$|\sin x| \le |x|$, where $0 \le x < 1$

$= |\sin\frac{1}{n}| \le |\frac{1}{n}|$

$K=1$, $\beta_n = \frac{1}{n}$

*by using formula *

$\therefore \lim_{n\to\infty} \sin\frac{1}{n} = O(\frac{1}{n})$

b. $\lim_{n\to\infty} \sin\frac{1}{n^2} = 0$

$|\sin\frac{1}{n^2}| \le |\frac{1}{n^2}|$

$K=1$, $\beta_n = \frac{1}{n^2}$

$\therefore \lim_{n\to\infty} \sin\frac{1}{n^2} = O(\frac{1}{n^2})$

c. $\lim_{n\to\infty}\left(\sin\frac{1}{n}\right)^2 = 0$

note: square and Absolute value can cancel each other

$= \left|\left(\sin\frac{1}{n}\right)^2\right| \le \left|\left(\frac{1}{n}\right)^2\right| = \left(\frac{1}{n^2}\right)$

$k=1$ ↘ $\quad$ ↳ B_n

$\therefore \lim_{n\to\infty}\left(\sin\frac{1}{n}\right)^2 = O\left(\frac{1}{n^2}\right)$

d. $\lim_{n\to\infty}[\ln(n+1) - \ln(n)] = 0$

to solve Q use ⇒ $|\ln(1+x)| \le |x|$, where $0 < x < 1$

$= [\ln(n+1) - \ln(n)] = \ln\frac{n+1}{n} = \ln\left(\frac{n}{n} + \frac{1}{n}\right) = \ln\left(1+\frac{1}{n}\right)$

$\Rightarrow \left|\ln\left(1+\frac{1}{n}\right)\right| \le 1\left|\left(\frac{1}{n}\right)\right|$

$k=1$ $\quad$ ↳ B_n

$\therefore \lim_{n\to\infty}[\ln(n+1) - \ln(n)] = O\left(\frac{1}{n}\right)$

$\ln(ab) = \ln(a) + \ln(b)$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$\ln(a^b) = b\ln(a)$

② Rate of Convergence for function ∴

formula ∴ $|F(h) - L| \le K|G(h)| \Rightarrow F(h) = L + O(G(h))$. where $G(h) = h^p$,

Maclurin Series ∴ $\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!} + \cdots$

$$\frac{e^h}{h} \quad \frac{}{h}$$

need the first two values

$$\frac{\cosh}{h} = \frac{1 - \frac{h^2}{2} + \frac{h^3}{3} + \cdots}{h}$$

$$\frac{e^h}{h} = \frac{1 + \frac{h^2}{2} + \frac{h^3}{3} + \cdots}{h}$$

7. Find the rates of convergence of the following functions as $h \to 0$.

a. $\lim_{h\to 0} \frac{\sin h}{h} = 1$ → L

$$= \frac{\sin h}{h} = \frac{h - \frac{h^3}{6} + \frac{h^5}{120}}{h} = \frac{h}{h} - \frac{\frac{h^3}{6}}{h} = 1 - \boxed{\frac{h^2}{6}}$$ use the first (h)

$$\text{so } \lim_{h\to 0} \frac{\sin h}{h} = L + O(h^p) = 1 + O(h^2)$$

b. $\lim_{h\to 0} \frac{1 - \cos h}{h} = 0$ → L

$\frac{\cosh}{h} = \frac{1 - \frac{h^2}{2} + \frac{h^3}{3} + \cdots}{h}$

$$\frac{1-\cos h}{h} = \frac{1-\left(1 - \frac{h^2}{2} + \frac{h^3}{3} + \cdots\right)}{h} = \frac{1 - 1 + \frac{h^2}{2} - \frac{h^3}{3} + \cdots}{h} = \frac{\frac{h^2}{2}}{h} - \frac{\frac{h^3}{3}}{h}$$

$$\frac{1-\cos h}{h} = \boxed{\frac{h}{2}} - \frac{h^2}{3}$$

use the first (h)

$\frac{0}{0}$ $\lim_{h\to 0} \frac{1-\cos h}{h} = 0 + O(h) = O(h)$

c. $\lim_{h\to 0} \frac{\sin h - h\cos h}{h} = 0$ → L

$\frac{\sin h - h\cos h}{h} = \frac{(h - \frac{h^3}{6} + \cdots) - h(1 - \frac{h^2}{2} + \cdots)}{h}$

$= \frac{(h - \frac{h^3}{6})(-h + \frac{h^3}{2})}{h} = \frac{-h^3}{6} + \frac{h^3}{2} \quad \frac{\times 3}{\times 3} = \frac{-h^3 + 3h^3}{6} = \boxed{\frac{2h^3}{6}}$

take (h^3) ↙

$\frac{0}{0}$ $\lim_{h\to 0} \frac{\sin h - h\cos h}{h} = 0 + O(h^3) = O(h^3)$

d. $\lim_{h\to 0} \frac{1 - e^h}{h} = -1$ → L

$\frac{e^h}{h} = \frac{1 + \frac{h^2}{2} + \frac{h^3}{3} + \cdots}{h}$

$= \frac{1 - e^h}{h} = \frac{1 - (1 + \frac{h^2}{2})}{h} = \frac{1 - 1 - \frac{h^2}{2}}{h} = \frac{\frac{-h^2}{2}}{h} = \boxed{\frac{-h}{2}}$ → take (h) w/o sign

$\frac{0}{0}$ $\lim_{h\to 0} \frac{1 - e^h}{h} = -1 + O(h)$